The Virahāṅka-Fibonacci and Related Sequences

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ABSTRACT

This note presents Virahāṅka's original proof of the sequence associated with prosody that is now known variously as the Virahāṅka-Fibonacci sequence, Fibonacci sequence, or just the Virahāṅka sequence. This sequence is also seen as the number of arrangements of beads in a necklace of a certain value, where each bead has the value of 1 or 2. This sequence was implicitly known as early as fourth to second century BCE in the work of Piṅgala, and was formally derived by Virahāṅka about 600 years before Fibonacci and, therefore, the last name is appropriate. The proof given here leads easily to the generalization of the Nārāyaṇa sequence.

Keywords: Virahāṅka-Fibonacci sequence, Fibonacci sequence, Virahāṅka numbers, Virahanka sequence, Narayana sequence

INTRODUCTION

Research has shown that reality is to be viewed as a whole and this wholeness gets expressed in terms of patterns that repeat at various scales as, for example, in the spiral patterns of the Whirlpool galaxy, the Nautilus shell, and the spiral aloe plant, and self-similar behavior as in the Romanesque broccoli and the Barnsley fern.

Figure 1. *Romanesco broccoli* (left); *spiral aloe* (right)

Some of these patterns emerge out of the optimality from the perspective of logic of $(1+1/x)^x$as x goes to infinity, which leads to the number e = 2.718281828.. This number appears is central to mathematical analysis and provides insight to our understanding of cosmology, theoretical physics, and efficient representation of data [1-4]. These patterns are associated with fractal behavior that has an evolutionary basis.

A related number is the solution to the equation $x = (1 + 1/x)$, which is the Golden Ratio, Φ, 1.618033989…[5]. It is found at the basis of stock-market data, petal patterns of flowers, and even the planet periods. When raised to the powers -3, -1, 0, 1, 5, 7, the Golden Ratio gives the periods of Mercury, Venus, Earth, Jupiter, and Saturn in years, indicating that the solar system must be viewed as a single whole.

Table 1. Planet periods

	Mercury	Venus	Earth	Jupiter	Saturn
Power of Φ	-3	-1	0	5	7
Decimal value	0.24	0.62	1.0	11.1	29.0
Actual period	0.24	0.62	1.0	11.9	29.5

The Golden Ratio is also obtained in the limit by dividing consecutive elements, the larger by the smaller, of the Virahāṅka (also known as the Fibonacci in the West) sequence: 1, 1, 2, 3, 5, 8, 13, 21… where the next term is the sum of the preceding two terms.

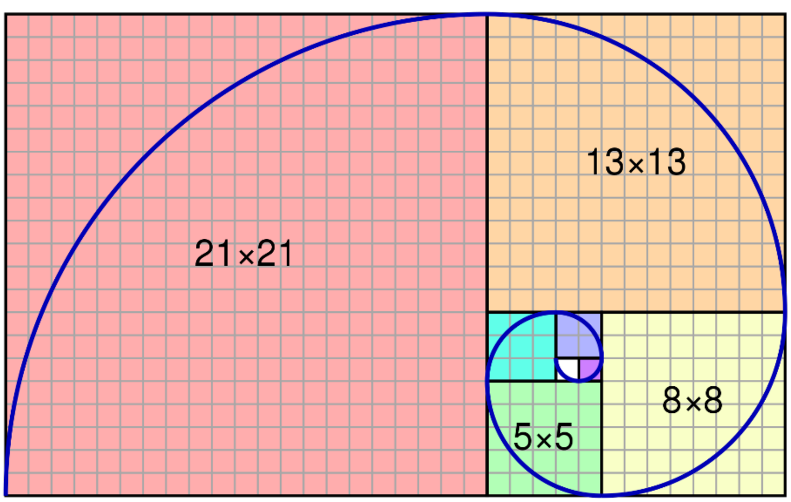

Figure 2. *The Virahāṅka numbers as a spiral*

The Virahāṅka sequence, described in the 6th century or 7th century, was introduced to Europe by Leonardo of Pisa, also called Fibonacci (1170-1250), in his book *Liber Abaci* in 1202. This book was essentially a translation of Indian mathematics that had come to him through Arabic reworkings of it.

In the Virahāṅka sequence 0, 1, 1, 2, 3, 5, 8, 13, . . ., the nth term is given by

$$V_n = V_{n-1} + V_{n-2}. \qquad (1)$$

Virahāṅka (विरहाङ्क) (*viraha* = separation, *aṅka* = mark) is believed to have lived in the 6th or 7th century. His work on prosody builds on the Chandaḥsūtra of Piṅgala (4th to second century BCE), and was the basis for the12th-century commentaries by Gopāla and Hemacandra Sūrī (1089-1172). Although the sequence is implicit in the Meru Prastāra of Piṅgala (see [6], which provides considerable historical background), it is reasonable to call the sequence after Virahāṅka since he explicitly describes it in his Prakrit work Vṛttajātisamuccya, वृत्तजातिसमुच्चय, and provides the justification.

The priority of Virahāṅka has been stressed by Donald Knuth in his historical surveys of combinatorics [7,8], and although the numbers come from poetry and music, one can also see them as related to the number of arrangements of beads in a necklace of a certain value where each individual bead has value of 1 or 2.

This paper provides the elegant proof given by Virahāṅka for the generation of the numbers of the sequence, and presents some additional implications.

NUMBER OF LAGHU AND GURU SYLLABLES

The mātrās are the morae of phonology, and often equal to syllables. Sanskrit prosody speaks of letters having a single mātrā called *laghu* (light) and those having two morae called *guru* (heavy). The former will be denoted by ℓ and the latter by g. Therefore, the weight of ℓ= 1 and that of g = 2.

Mātrā-vṛttas are meters in which the number of morae remains constant and the number of letters is arbitrary. The expansion in terms of the *laghu* and the *guru* is called a *prastāra* and it is shown as a table or matrix.

Theorem (Virahāṅka):

दो दो पुव्वविअप्पे जा मेलविऊण जायए संखा
सा उत्तरमत्ताणं संखाए एस निद्देसो। (Prakrit)

द्वौ द्वौ पूर्वविकल्पौ या मेलयित्वा जायते सङ्ख्या।
सा उत्तरमात्राणां सङ्ख्याया एष निर्देशः॥ (Sanskritized)
(वृत्तजातिसमुच्चयः ६.४९)

The next number is created by joining the previous two [numbers]
That indicates the count of the total mātrās.

Proof. A single mātrā will be represented by the single case of:

ℓ

The case of two mātrās has two arrangements as shown:

g
$\ell\ell$

One can easily construct prastāras for mātrās ranging from 1 to 5 as shown in Table 2. These represent all combinations of the mātrās ℓ and g to give the correct weight for the column.

Table 2. The prastāra for mātrās ranging from 1 to 5

1 mātrā	2 mātrās	3 mātrās	4 mātrās	5 mātrās
ℓ	g	$\ell\ g$	$g\ g$	$\ell\ g\ g$
	$\ell\ \ell$	$g\ \ell$	$\ell\ell\ g$	$g\ \ell\ g$
		$\ell\ \ell\ \ell$	$\ell\ g\ \ell$	$\ell\ell\ \ell\ g$
			$g\ \ell\ \ell$	$gg\ \ell$
			$\ell\ \ell\ \ell\ \ell$	$\ell\ \ell\ g\ \ell$
				$\ell g\ell\ \ell$
				$g\ \ell\ \ell\ \ell$
				$\ell\ \ell\ \ell\ \ell\ \ell$

It is quite clear that since all the combinations of, say, of 3 and 4 mātrās are given in the third and fourth columns, one needs to associate the suffix (or the prefix) of g to each of the entries in the third column (3+2 =5) and that of ℓ to each of the entries of the fourth column (4+1=5) to provide the complete prastāra of the fifth column.

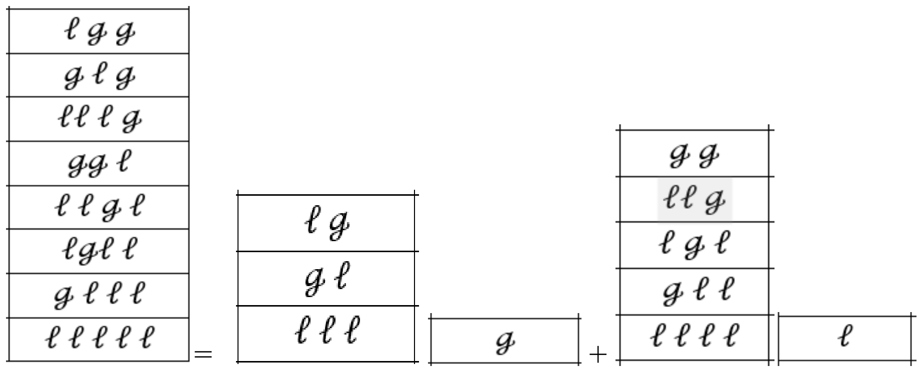

Figure 3. *The basis of the recurrence relation*

In terms of numbers 8 = 3 +5. The expansion of mātrā- vṛttas corresponds to a partitioning of a number (the count of morae in the meter), where the digits take on the values 2 and 1 and their order is relevant.

The generalization of the construction of Figure 3 proves that the count in the *n*th column will equal the counts of the *n*-1 and *n*-2 columns.■

Therefore, from this theorem, we get the sequence:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, and so on.

This Virahāṅka sequence, if extended to the left to include zero, gets us:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89. . . (2)

An interesting property of the Virahāṅka sequence is that one can see it also as a power sequence where the next element is two times the previous element together with subtractions which in themselves constitute another similar sequence. Thus we can generate the elements 1 1 2 3 5 8 13 21 34 55 89…. recursively in the following manner:

$1 \times 2 - 1 = 1,\ 1 \times 2 + 0 = 2,\ 2 \times 2 - 1 = 3,\ 3 \times 2 - 1 = 5,\ 5 \times 2 -$

$2 = 8,\ 8 \times 2 - 3 = 13,\ 13 \times 2 - 5 = 21,\ 21 \times 2 - 8 = 34,\ 34 \times 2 - 13 = 55$, and so on.

One observes the sequence of numbers that is added (or subtracted) after multiplication by 2, is the sequence

-1, 0, -1, -1, -2, -3, -5, -8, -13, and so on.

which is a Virahāṅka sequence.

BEADS IN A NECKLACE

The sequence could also be viewed as the number of arrangements of two kinds of beads in a necklace of a certain value, where each bead has the value of 1 or 2.

As a straightforward extension, consider the problem of finding arrangements of beads of three different kind (colors) for a given total value where the cost of the individual beads is 1, 2, and 3 units.

Writing the prastāras horizontally, where the cost of the bead as number represents the corresponding symbol, we have:

$N_2 = 2$; the members are 2, 1 1
$N_3 = 4$; the members are 3, 2 1, 1 2, 1 1 1
$N_4 = 7$; the members are 3 1, 1 3, 2 2, 2 1 1, 1 2 1, 1 1 2, 1 1 1 1

$N_5 = 7+4+2 = 13$ which is obtained by concatenating 3 to elements corresponding to N_2, 2 to elements corresponding to N_3, and 1 to elements corresponding to N_4. This set is: 2 3, 1 1 3, 3 2, 2 1 2, 1 2 2, 1 1 1 2, 3 1 1, 1 3 1, 2 2 1, 2 1 1 1, 1 2 1 1, 1 1 2 1, 1 1 1 1 1, that completely enumerates all sequences of 1, 2, and 3 that add up to 5.

This may be generalized to necklaces of a given value, constructed using *k* different beads of value 1, 2, 3, .., *k* or a subset thereof. This will lead to a sequence where the next term is the sum of the preceding *k* terms or a subset thereof as is shown below.

RELATED SEQUENCES

Since Gopāla and Hemacandra did considerable work on the numbers prior to Fibonacci [9], it has been suggested that the name Gopāla-Hemacandra numbers be used for the general sequence:

$$a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ \ldots \tag{3}$$

for any pair *a, b*, which for the case $a = 1$, $b = 1$ represents the Virahāṅka numbers.

Nārāyaṇa Paṇḍita's book Gaṇita Kaumudi (1356) studied the sequence related to the following problem [10]: *A cow gives birth to a calf every year. In turn, the calf gives birth to another calf when it is three years old. What is the number of progeny produced during twenty years by one cow?*

The *n*th term of the Nārāyaṇa series is defined by:

$$N_n = N_{n-1} + N_{n-3}. \tag{4}$$

The sequence numbers are: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, …

The Nārāyaṇa series may be cast in form to the following necklace problem: *What are the number of arrangements of beads in a necklace of value n, where the beads have values of 1 and 3?*

Clearly, the arrangements are:

1: 1 = 1
2: 1 = 1 1
3: 2 = 1 1 1, 3
4: 3 = 3 1, 1 3, 1 1 1 1
5: 4 = 3 1 1, 1 3 1, 1 1 3, 1 1 1 1 1
6: 6 = 3 3, 3 1 1 1, 1 3 1 1, 1 1 3 1, 1 1 1 3, 1 1 1 1 1 1
7: 9 = 3 3 1, 3 1 3, 1 3 3, 3 1 1 1 1, 1 3 1 1 1, 1 1 3 1 1, 1 1 1 3 1,
1 1 1 1 3, 1 1 1 1 1 1 1

Thus we get the sequence 1, 1, 2, 3, 4, 6, 9, ..

An application of the Nārāyaṇa number for universal coding has been proposed [11].

CONCLUSIONS

This note is to make accessible the explanation advanced by Virahāṅka as rationale for his sequence in the context of prosody. This sequence is also seen as the number of arrangements of beads in a necklace of a certain value, where each bead has the value of 1 or 2. It is shown that the construction of Virahāṅka can be easily generalized to other sequences such as the Nārāyaṇa sequence.

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